Navier – Stokes Millennium Prize Problem. Alternative Solution

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The Official Millennium Prize Problem Description requires a proof of "existence and smoothness of Navier – Stokes (NSE) solutions". Other possible approach (as "Strategy 1" of three) is considered by Terence Tao in his blog. It is "…an exact, explicit transformation to a significantly simpler PDE or ODE ". Such solution of the NSE Problem is proposed in this article. Two alternatives of exact, explicit NSE transformation are proposed.

Key Words and Phrases: Navier – Stokes Equations, vorticity transport equations, incompressible fluid, volume deformation, velocity of volume deformation, acceleration of volume deformation, acceleration divergence, partial derivatives, composite function, vector lines equations.

2000 Mathematics Subject Classifications: 35Q30, 76D05

1. Introduction

The Navier–Stokes equations (NSE) are given by

$$\rho\vec{F} - \operatorname{grad} p + \mu\nabla^2\dot{\vec{u}} = \rho\ddot{\vec{u}} \tag{1}$$

with such equation is called a continuity equation for incompressible fluids [1, p. 174]

$$\operatorname{div}\dot{\vec{u}} = \frac{\partial \dot{u}_x}{\partial x} + \frac{\partial \dot{u}_y}{\partial y} + \frac{\partial \dot{u}_z}{\partial z} = 0\,.$$

Here, $\vec{F}$ - vector unit mass force, p - pressure, $\dot{\vec{u}}$ - velocity vector, $\ddot{\vec{u}} = d\dot{\vec{u}}/dt$ - acceleration vector, ρ - density, μ - viscosity, ∇^2 - Laplace operator.

We now consider two possible approaches of an exact NSE transformation.

First approach. We can eliminate the pressure p by taking an operator rot (alternative notation curl) of both sides of equations (1). In this case equations

(1) reduce to the Vorticity transport equations. In two dimensions (2D), these equations are well known [2, p. 531; 3, p. 74; 4, p. 321]

$$\nu\nabla^2\Omega = \frac{d\Omega}{dt}, \quad 2\Omega = \mathrm{rot}\,\dot{\vec{u}}, \quad \nu = \frac{\mu}{\rho}. \tag{2}$$

In three dimensions (3D), it is known for a long time that Vorticity transport equations have additional terms [2, p. 531; 4, p. 294]

$$\nu\nabla^2\Omega + (\Omega\cdot\nabla)\dot{\vec{u}} = \frac{d\Omega}{dt}. \tag{2*}$$

However, why 1D, 2D and 3D Navier - Stokes equations in the vector form are identical? In that case, probably, the vorticity transport equations must be identical too. But this conjecture requires the proof. We can show that 3D vorticity transport equations look like (2).

Second approach. After taking an operator div of both sides of equations (1) the NSE becomes [5, p. 74]

$$\rho\,\mathrm{div}\,\vec{F} - \mathrm{div}\,\mathrm{grad}\,p + \mu\nabla^2\,\mathrm{div}\,\dot{\vec{u}} = \rho\,\mathrm{div}\,\ddot{\vec{u}}. \tag{1*}$$

Here, $\mu\nabla^2\,\mathrm{div}\,\dot{\vec{u}} = 0$, $\mathrm{div}\,\mathrm{grad}\,p = \nabla^2 p$ [6, p. 171]. Therefore equation (1*) can be written as

$$\nabla^2 p = -\rho\,\mathrm{div}(\ddot{\vec{u}} - \vec{F}). \tag{1**}$$

Authors of [5] has obtained equation (1**) if $\vec{F} = 0$ and $\dot{\vec{u}}\,(\ddot{\vec{u}})$ is a given function. However these authors could not prolong transformation because they did not consider such analogy [1, p. 107; 7, p. 329].

$$\varepsilon_o = \frac{1}{\delta V} d(\delta V) = \mathrm{div}\,\vec{u}, \quad \dot{\varepsilon}_o = \frac{1}{\delta V}\frac{d(\delta V)}{dt} = \mathrm{div}\,\dot{\vec{u}}. \tag{3}$$

Here, ε_o - volume deformation, $\vec{u}$ - infinitesimal displacement vector ($\vec{u}$ - any displacement vector), $\dot{\varepsilon}_o$-velocity of volume deformation. By analogy, the acceleration divergence $\mathrm{div}\,\ddot{\vec{u}}$ can be written as ($\ddot{\varepsilon}_o$ - acceleration of volume deformation)

$$\ddot{\varepsilon}_o = \frac{1}{\delta V}\frac{d^2(\delta V)}{dt^2} = \mathrm{div}\,\ddot{\vec{u}}. \tag{3*}$$

The concept “acceleration divergence” does not meet in scientific publications. At the same time, the concepts div$\vec{u}$ and div$\dot{\vec{u}}$ are fundamental in continuum mechanics, vector calculus, etc. However, this optimistic conjecture $\ddot{\varepsilon}_o = \mathrm{div}\,\ddot{\vec{u}}$ requires the proof. Such proof received by several alternative methods. Here we propose the simplest method which allows to implement both above approaches.

2. Transformation of 3D vorticity transport equations

Let's consider the expressions $2\ddot{\Omega}_i = \mathrm{rot}_i\,\ddot{\vec{u}}$ which can be written in a general form

$$\mathrm{rot}_x\,\ddot{\vec{u}} = \frac{\partial \ddot{u}_z}{\partial y} - \frac{\partial \ddot{u}_y}{\partial z},\quad \mathrm{rot}_y\,\ddot{\vec{u}} = \frac{\partial \ddot{u}_x}{\partial z} - \frac{\partial \ddot{u}_z}{\partial x},\quad \mathrm{rot}_z\,\ddot{\vec{u}} = \frac{\partial \ddot{u}_y}{\partial x} - \frac{\partial \ddot{u}_x}{\partial y} \tag{4}$$

After let's consider the expression of one component $\mathrm{rot}_x\,\ddot{\vec{u}}$. Each accelerations vector components can be represented in such expanded form [1, p. 39]

$$\ddot{u}_x = \frac{d\dot{u}_x}{dt} = \frac{\partial \dot{u}_x}{\partial t} + \dot{u}_x\frac{\partial \dot{u}_x}{\partial x} + \dot{u}_y\frac{\partial \dot{u}_x}{\partial y} + \dot{u}_z\frac{\partial \dot{u}_x}{\partial z},$$
$$\ddot{u}_y = \frac{d\dot{u}_y}{dt} = \frac{\partial \dot{u}_y}{\partial t} + \dot{u}_x\frac{\partial \dot{u}_y}{\partial x} + \dot{u}_y\frac{\partial \dot{u}_y}{\partial y} + \dot{u}_z\frac{\partial \dot{u}_y}{\partial z},$$
$$\ddot{u}_z = \frac{d\dot{u}_z}{dt} = \frac{\partial \dot{u}_z}{\partial t} + \dot{u}_x\frac{\partial \dot{u}_z}{\partial x} + \dot{u}_y\frac{\partial \dot{u}_z}{\partial y} + \dot{u}_z\frac{\partial \dot{u}_z}{\partial z}.$$

Let's differentiate the components $\ddot{u}_z$, $\ddot{u}_y$

$$\frac{\partial \ddot{u}_z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \dot{u}_z}{\partial t} + \dot{u}_x\frac{\partial \dot{u}_z}{\partial x} + \dot{u}_y\frac{\partial \dot{u}_z}{\partial y} + \dot{u}_z\frac{\partial \dot{u}_z}{\partial z}\right) =$$
$$= \frac{\partial^2 \dot{u}_z}{\partial y\partial t} + \dot{u}_x\frac{\partial^2 \dot{u}_z}{\partial y\partial x} + \frac{\partial \dot{u}_x}{\partial y}\frac{\partial \dot{u}_z}{\partial x} + \dot{u}_y\frac{\partial^2 \dot{u}_z}{\partial y^2} + \frac{\partial \dot{u}_y}{\partial y}\frac{\partial \dot{u}_z}{\partial y} + \dot{u}_z\frac{\partial^2 \dot{u}_z}{\partial y\partial z} + \frac{\partial \dot{u}_z}{\partial y}\frac{\partial \dot{u}_z}{\partial z},$$
$$\frac{\partial \ddot{u}_y}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial \dot{u}_y}{\partial t} + \dot{u}_x\frac{\partial \dot{u}_y}{\partial x} + \dot{u}_y\frac{\partial \dot{u}_y}{\partial y} + \dot{u}_z\frac{\partial \dot{u}_y}{\partial z}\right) =$$
$$= \frac{\partial^2 \dot{u}_y}{\partial z\partial t} + \dot{u}_x\frac{\partial^2 \dot{u}_y}{\partial z\partial x} + \frac{\partial \dot{u}_x}{\partial z}\frac{\partial \dot{u}_y}{\partial x} + \dot{u}_y\frac{\partial^2 \dot{u}_y}{\partial z\partial y} + \frac{\partial \dot{u}_y}{\partial z}\frac{\partial \dot{u}_y}{\partial y} + \dot{u}_z\frac{\partial^2 \dot{u}_y}{\partial z^2} + \frac{\partial \dot{u}_z}{\partial z}\frac{\partial \dot{u}_y}{\partial z}. \tag{5}$$

The difference between these expressions can be represented as

$$\frac{1}{2}\mathrm{rot}_x\ddot{\vec{u}} = \bar{\Omega}_x = \frac{\partial\Omega_x}{\partial t} + \dot{u}_x\frac{\partial\Omega_x}{\partial x} + \dot{u}_y\frac{\partial\Omega_x}{\partial y} + \dot{u}_z\frac{\partial\Omega_x}{\partial z} +$$
$$+\Omega_x\frac{\partial\dot{u}_y}{\partial y} + \Omega_x\frac{\partial\dot{u}_z}{\partial z} + \frac{1}{2}\left(\frac{\partial\dot{u}_x}{\partial y}\frac{\partial\dot{u}_z}{\partial x} - \frac{\partial\dot{u}_x}{\partial z}\frac{\partial\dot{u}_y}{\partial x}\right). \quad (5^*)$$

Other expressions ($\mathrm{rot}_y\ddot{\vec{u}}$, $\mathrm{rot}_z\ddot{\vec{u}}$) can be written by analogy.

For 2D flow $\dot{u}_x = 0$. Therefore

$$\frac{1}{2}\mathrm{rot}_i\ddot{\vec{u}} = \bar{\Omega}_i = \frac{d\Omega_i}{dt} + \Omega_i\,\mathrm{div}\dot{\vec{u}}. \quad (5^{**})$$

Two terms in brackets of (5*), probably, can be written as $\Omega_x\frac{\partial\dot{u}_x}{\partial x}$. In this case last three terms of (5*) can be represented as $\Omega_x\,\mathrm{div}\dot{\vec{u}}$. However, this hypothesis requires the proof.

2.1. Hypothesis proof

To prove expressions (3*) and (5**) we use partial derivatives of a composite function, which can be given any number of auxiliary variables [8, p. 644]. Let $\dot{\vec{u}} = \dot{\vec{u}}(x,y,z,t)$ be a composite function. Let's fix a time $t = \bar{t}$. Suppose the velocity vector $\dot{\vec{u}} = \dot{\vec{u}}(x,y,z,t)$ can be represented by one auxiliary variable $\varsigma = \varsigma(x,y,z)$ as $\dot{\vec{u}} = \dot{\vec{u}}(\varsigma)$. We take into account that basic properties of the derivatives are maintained for the vectors [6, p. 79; 8, p. 516]. Then

$$\frac{\partial\dot{\vec{u}}}{\partial x_i} = \frac{\partial\dot{\vec{u}}}{\partial\varsigma}\frac{\partial\varsigma}{\partial x_i}, (x_i = x, y, z). \quad (6)$$

Formulas (6) can be written explicitly concerning a common factor $\frac{\partial\dot{\vec{u}}}{\partial\varsigma}$. This factor can be deleted. As a result we have

$$\frac{\partial\dot{\vec{u}}}{\partial x_i} = \frac{\partial\dot{\vec{u}}}{\partial x_j}\frac{\partial\varsigma/\partial x_i}{\partial\varsigma/\partial x_j}\ . \quad (7)$$

In component form formulas (7) look like

$$\frac{\partial \dot{u}_x}{\partial x_i} = \frac{\partial \dot{u}_x}{\partial x_j}\frac{\partial \varsigma/\partial x_i}{\partial \varsigma/\partial x_j},\ \frac{\partial \dot{u}_y}{\partial x_i} = \frac{\partial \dot{u}_y}{\partial x_j}\frac{\partial \varsigma/\partial x_i}{\partial \varsigma/\partial x_j},\ \frac{\partial \dot{u}_z}{\partial x_i} = \frac{\partial \dot{u}_z}{\partial x_j}\frac{\partial \varsigma/\partial x_i}{\partial \varsigma/\partial x_j} \qquad (8)$$

Relations (8) can be written explicitly concerning a common factor $\frac{\partial \varsigma/\partial x_i}{\partial \varsigma/\partial x_j}$. This factor can be deleted, if we will equate right-hand sides of these transformed expressions. Therefore

$$\frac{\partial \dot{u}_x}{\partial x_i}\frac{\partial \dot{u}_y}{\partial x_j} = \frac{\partial \dot{u}_x}{\partial x_j}\frac{\partial \dot{u}_y}{\partial x_i},\ \frac{\partial \dot{u}_x}{\partial x_i}\frac{\partial \dot{u}_z}{\partial x_j} = \frac{\partial \dot{u}_x}{\partial x_j}\frac{\partial \dot{u}_z}{\partial x_i},\ \frac{\partial \dot{u}_y}{\partial x_i}\frac{\partial \dot{u}_z}{\partial x_j} = \frac{\partial \dot{u}_y}{\partial x_j}\frac{\partial \dot{u}_z}{\partial x_i}\ . \qquad (8^*)$$

Now we must to prove the representation possibility $\dot{u}_i = \dot{u}_i(\varsigma)$. Such proof can be obtained from the vector lines equations (streamlines) [7, p. 318; 9, p. 155]

$$\frac{dx}{\dot{u}_x} = \frac{dy}{\dot{u}_y} = \frac{dz}{\dot{u}_z} = d\varsigma\ . \qquad (9)$$

Here, the symbols d denote differentials for $t = \bar{t}$.

Equations (9) can now be written in the form of integrals

$$\varsigma = \int \frac{1}{\dot{u}_x} dx + C_x \Rightarrow \varsigma = F_x(x, y, z, \bar{t}),$$

$$\varsigma = \int \frac{1}{\dot{u}_y} dy + C_y \Rightarrow \varsigma = F_y(x, y, z, \bar{t}),$$

$$\varsigma = \int \frac{1}{\dot{u}_z} dz + C_z \Rightarrow \varsigma = F_z(x, y, z, \bar{t}).$$

Three equivalent expressions $\varsigma = F_i(x, y, z, \bar{t})$ can be consider as algebraic system of equations with three unknown x, y, z. Note that $t = \bar{t}$ is a parameter. This system can be solved for x, y, z. As a result we will obtain $x = x(\varsigma, \bar{t}),\ y = y(\varsigma, \bar{t}),\ z = z(\varsigma, \bar{t})$. After substitution of these expressions into $\dot{\vec{u}} = \dot{\vec{u}}(x, y, z, \bar{t})$ we will obtain $\dot{u}_i = \dot{u}_i(\varsigma, \bar{t})$. Therefore, for $t = \bar{t}$ we actually have $\dot{u}_i = \dot{u}_i(\varsigma)$.

Now we will muster a possibility of the conjecture for formula (5*)

$$\frac{\partial \dot{u}_x}{\partial y}\frac{\partial \dot{u}_z}{\partial x} - \frac{\partial \dot{u}_x}{\partial z}\frac{\partial \dot{u}_y}{\partial x} = \Omega_x \frac{\partial \dot{u}_x}{\partial x}\ . \qquad (10)$$

To prove this formula we will write relations (8*) and any equalities (implying from these relations)

$$\frac{\partial\dot u_x}{\partial x_i}\frac{\partial\dot u_y}{\partial x_j}=\frac{\partial\dot u_x}{\partial x_j}\frac{\partial\dot u_y}{\partial x_i}\Rightarrow\frac{\partial\dot u_x}{\partial z}\frac{\partial\dot u_y}{\partial x}=\frac{\partial\dot u_x}{\partial x}\frac{\partial\dot u_y}{\partial z},$$
$$\frac{\partial\dot u_x}{\partial x_i}\frac{\partial\dot u_z}{\partial x_j}=\frac{\partial\dot u_x}{\partial x_j}\frac{\partial\dot u_z}{\partial x_i}\Rightarrow\frac{\partial\dot u_x}{\partial y}\frac{\partial\dot u_z}{\partial x}=\frac{\partial\dot u_x}{\partial x}\frac{\partial\dot u_z}{\partial y},$$
$$\frac{\partial\dot u_y}{\partial x_i}\frac{\partial\dot u_z}{\partial x_j}=\frac{\partial\dot u_y}{\partial x_j}\frac{\partial\dot u_z}{\partial x_i}. \tag{11}$$

The equalities after sign $\Rightarrow$ confirm a validity of (10). Therefore (5*) becomes

$$\frac{1}{2}\mathrm{rot}_x\ddot{\vec u}=\bar\Omega_x=\frac{\partial\Omega_x}{\partial t}+\dot u_x\frac{\partial\Omega_x}{\partial x}+\dot u_y\frac{\partial\Omega_x}{\partial y}+\dot u_z\frac{\partial\Omega_x}{\partial z}+\Omega_x\left(\frac{\partial\dot u_x}{\partial x}+\frac{\partial\dot u_y}{\partial y}+\frac{\partial\dot u_z}{\partial z}\right). \tag{12}$$

Thus (12) coincides with (5**).

The expressions for two remaining components can be written analogously. For incompressible fluid we have $\mathrm{div}\dot{\vec u}=0$. Therefore vorticity transport equations for 3D flow take the form (2). In scalar form we have three equations

$$\nu\nabla^2\Omega_i=\frac{d\Omega_i}{dt},\quad (i=x,y,z). \tag{13}$$

After taking an operator div of both sides of equations (13) we obtain

$$\nu\nabla^2\,\mathrm{div}\,\Omega=\mathrm{div}\left(\frac{d\Omega}{dt}\right). \tag{14}$$

Here $\mathrm{div}\,2\Omega=\mathrm{div}\,\mathrm{rot}\,\dot{\vec u}=0$. Therefore

$$\mathrm{div}\left(\frac{d\Omega}{dt}\right)=0. \tag{15}$$

2.2. Exact transformation of (13)

Now consider two possible approaches of an exact transformation of (13).

First approach.Let's take $\bar\Omega_i=\dfrac{d\Omega_i}{dt}$ as an unknown quantity (i.e.$\Omega_i=\int\bar\Omega_i dt$). Then equations (13) can be written as

$$\nu\nabla^2\int\bar{\Omega}_i dt=\bar{\Omega}_i,\quad i=x,y,z\ . \tag{16}$$

The simplifying consequence $\bar{\Omega}_i=0$ follows from equations (16) for an ideal fluid ($\nu=0$).

Second approach. Equations (13) can be written as

$$\nu\nabla^2\Omega_i=\frac{\partial\Omega_i}{\partial t}+\dot{u}_x\frac{\partial\Omega_i}{\partial x}+\dot{u}_y\frac{\partial\Omega_i}{\partial y}+\dot{u}_z\frac{\partial\Omega_i}{\partial z}, \tag{13*}$$

In the case $\mathrm{div}\dot{\vec{u}}=0$ formula for a vectors field restoration has such form [9, p. 243]

$$\dot{\vec{u}}=\frac{1}{4\pi}\mathrm{rot}\iiint\frac{\mathrm{rot}\,\dot{\vec{u}}}{r}dV,\quad (\mathrm{rot}\,\dot{\vec{u}}=2\Omega)\,.$$

After substitution of $\dot{u}_i$ into formula (13*) we will obtain three equations for Ω_i. However this approach, probably, is hopeless.

3. Exact transformation of (1**)

Let's calculate $\mathrm{div}\ddot{\vec{u}}$. Then after some transformations

$$\mathrm{div}\ddot{\vec{u}}=\frac{\partial\ddot{u}_x}{\partial x}+\frac{\partial\ddot{u}_y}{\partial y}+\frac{\partial\ddot{u}_z}{\partial z}=\frac{\partial}{\partial t}\mathrm{div}\dot{\vec{u}}+\dot{u}_x\frac{\partial}{\partial x}\mathrm{div}\dot{\vec{u}}+\dot{u}_y\frac{\partial}{\partial y}\mathrm{div}\dot{\vec{u}}+\dot{u}_z\frac{\partial}{\partial z}\mathrm{div}\dot{\vec{u}}+$$
$$+\left[\left(\frac{\partial\dot{u}_x}{\partial x}\right)^2+\left(\frac{\partial\dot{u}_y}{\partial y}\right)^2+\left(\frac{\partial\dot{u}_z}{\partial z}\right)^2+2\left(\frac{\partial\dot{u}_x}{\partial y}\frac{\partial\dot{u}_y}{\partial x}+\frac{\partial\dot{u}_y}{\partial z}\frac{\partial\dot{u}_z}{\partial y}+\frac{\partial\dot{u}_x}{\partial z}\frac{\partial\dot{u}_z}{\partial x}\right)\right]. \tag{17}$$

This formula can be written as

$$\mathrm{div}\ddot{\vec{u}}=\frac{d}{dt}\mathrm{div}\dot{\vec{u}}+(\mathrm{div}\dot{\vec{u}})^2$$

if next equality is true

$$\left(\frac{\partial\dot{u}_x}{\partial x}\right)^2+\left(\frac{\partial\dot{u}_y}{\partial y}\right)^2+\left(\frac{\partial\dot{u}_z}{\partial z}\right)^2+2\left(\frac{\partial\dot{u}_x}{\partial y}\frac{\partial\dot{u}_y}{\partial x}+\frac{\partial\dot{u}_y}{\partial z}\frac{\partial\dot{u}_z}{\partial y}+\frac{\partial\dot{u}_x}{\partial z}\frac{\partial\dot{u}_z}{\partial x}\right)=(\mathrm{div}\dot{\vec{u}})^2\,. \tag{17*}$$

The realization of (17*) require next equality :

$$\frac{\partial \dot{u}_x}{\partial y}\frac{\partial \dot{u}_y}{\partial x}+\frac{\partial \dot{u}_y}{\partial z}\frac{\partial \dot{u}_z}{\partial y}+\frac{\partial \dot{u}_x}{\partial z}\frac{\partial \dot{u}_z}{\partial x}=\frac{\partial \dot{u}_x}{\partial x}\frac{\partial \dot{u}_y}{\partial y}+\frac{\partial \dot{u}_y}{\partial y}\frac{\partial \dot{u}_z}{\partial z}+\frac{\partial \dot{u}_x}{\partial x}\frac{\partial \dot{u}_z}{\partial z}. \qquad (17^{**})$$

Let's substitute $x_i = x, y, z, \ (x_j \neq x_i)$ into (8*). Then

$$\frac{\partial \dot{u}_x}{\partial x}\frac{\partial \dot{u}_y}{\partial y}=\frac{\partial \dot{u}_x}{\partial y}\frac{\partial \dot{u}_y}{\partial x}, \frac{\partial \dot{u}_x}{\partial x}\frac{\partial \dot{u}_z}{\partial z}=\frac{\partial \dot{u}_x}{\partial z}\frac{\partial \dot{u}_z}{\partial x}, \frac{\partial \dot{u}_y}{\partial y}\frac{\partial \dot{u}_z}{\partial z}=\frac{\partial \dot{u}_y}{\partial z}\frac{\partial \dot{u}_z}{\partial y}. \qquad (18)$$

This is a necessary condition for equality (17*). Therefore $\operatorname{div}\ddot{\vec{u}}=0$ if $\operatorname{div}\dot{\vec{u}}=0$.

Important note. Note that equality (17**) makes sense only for $\operatorname{rot}\dot{\vec{u}}\neq 0$. In the case $\operatorname{rot}\dot{\vec{u}}=0,\ \operatorname{div}\dot{\vec{u}}=0$ all terms of left-hand side (17*) are positive, and $\dot{\vec{u}}=\operatorname{grad}\varphi,\ \nabla^2\varphi=0$. The requirements $\operatorname{rot}\dot{\vec{u}}=0,\ \operatorname{div}\dot{\vec{u}}=0$ and (18) are unusual rigid and cannot be implemented in unison. This requirements does not allow to fulfill even such traditional boundary conditions as a fluid adhesion. The situation becomes quite explainable if to pay attention to the following. The component-wise entry of $\operatorname{rot}\dot{\vec{u}}=0,\ \operatorname{div}\dot{\vec{u}}=0$ and (17**) give the strongly redefined system of equations [10, p. 28]. This system consists of five differential equations with only three unknown functions $\dot{u}_x, \dot{u}_y, \dot{u}_z$. However this will be the object of another paper.

Taking into account that $\operatorname{div}\ddot{\vec{u}}=0$, formulas (1**) can be conversed to a Poisson equation

$$\nabla^2 p = \rho \operatorname{div}\vec{F}.$$

Therefore $\nabla^2 p = 0$ if $\operatorname{div}\vec{F}=0$. In that case we can eliminate pressure p by taking ∇^2 of both sides of equations (1). After such transformation the NSE becomes

$$\nabla^2\left(\mu\nabla^2\dot{u}_i-\rho\ddot{u}_i\right)=0. \qquad (19)$$

Using the notation

$$\dot{u}_i=\int \ddot{u}_i dt, \qquad (20)$$

we find the new general equations for incompressible fluid

$$\nabla^2 p = 0,\ \ \nabla^2\left(\nu\nabla^2\int \ddot{u}_i dt-\ddot{u}_i\right)=0,\ \ \nu=\frac{\mu}{\rho},\ \ (i=1,2,3). \qquad (21)$$

Each equation (21) include only one from four unknown functions $p, \ddot{u}_x, \ddot{u}_y, \ddot{u}_z$.

For the partial solutions of (21) it is possible to use a traditional boundary conditions (a boundary adhesion). The acceleration vector $\ddot{\vec{u}}$, as well as a velocity vector $\dot{\vec{u}}$, is equal to null on the immobile boundary.

Important note. The important consequence $\nabla^2 \ddot{u}_i = 0$ implies from equations (21) for ideal fluid ($\nu = 0$). This result allows to understand why the solutions of the Euler equations do not satisfy traditional boundary conditions (we should recall the property of harmonic functions about extremum). More profound conclusions can be obtained if we use the representation of general solution of harmonic functions [11, p. 58].

4. Conclusion

The well-known 3D and 2D vector Vorticity transport equations are different [2, p. 531; 3, p. 74; 4, p. 294, 321]. In this paper we prove that 3D and 2D equations must be identical, and therefore 3D equations can be conversed to a traditional form (2).

First important consequence $\frac{d\Omega}{dt} = 0$ implies from equations (2) for ideal fluid ($\nu = 0$). This consequence allows to simplify the classical solutions of 3D vortex motions of ideal medium. First of all the Helmholtz equation and theorems are subject to correction [1, p. 332; 2, p. 115].

Second important consequence $\nabla^2 \ddot{u}_i = 0$ implies from equations (19) if $\nu = 0$. In this motive Charles Fefferman (author of the Official Millennium Problem Description) has noted: "*…problems are also open and very important for the Euler equations (*$\nu = 0$*), although the Euler equation is not on the Clay Institute's list of prize problems*".

The precise Official statement of the problem, given by the Clay Mathematics Institute requires the proof of *"existence and smoothness of Navier - Stokes solutions"*. The most effective way of such proof is correct NSE transformation to more simple equations. The proof in such cases should not cause serious difficulties. Such approach (as "Strategy 1" from three) is considered by Terence Tao in his blog: "*…exact and explicit solutions (or at least an* ***exact, explicit transformation to a significantly simpler PDE or ODE****)…*"[12]. Such method is proposed in this article. Equations (16) and (21) are two alternatives of such exact, explicit transformation. The explicit NSE solution is proposed in article [13]. However this solution requires checkout of correspondence at least to the formula $\nabla^2 p = \rho \operatorname{div} \vec{F}$ or $\nabla^2 p = 0$.

The abstract of this paper is published in [14, p. 197-198].

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