Vector PDE Paradoxes

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In this paper the additional equalities (differential equations) are proposed for the vector PDE. These equations are obligatory requirements (properties) of three functions forming a vector field on Euclidean space. Therefore all solutions of the vector PDE should satisfy these requirements. Otherwise without these requirements, all so called “exact solutions” are not solutions of the vector PDE.

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1. Introduction

Let’s consider the introduction in this problem on an example of well-known Navier–Stokes equations (NSE)

$$\rho\vec{F} - \operatorname{grad} p + \mu\nabla^2\dot{\vec{u}} = \rho\ddot{\vec{u}} \tag{1}$$

with a continuity equation for incompressible fluids [1, p. 174]

$$\operatorname{div}\dot{\vec{u}} = \frac{\partial \dot{u}_x}{\partial x} + \frac{\partial \dot{u}_y}{\partial y} + \frac{\partial \dot{u}_z}{\partial z} = 0\,. \tag{2}$$

Here, $\vec{F}$ - vector of a given, externally applied force (e.g. gravity), $\dot{\vec{u}}$ - velocity vector, $\ddot{\vec{u}} = d\dot{\vec{u}}/dt$ - acceleration vector, p - pressure (scalar), ρ - density (scalar), μ - viscosity (scalar), ∇^2 - Laplace operator.

The component form of (1) can be written so

$$\rho F_x - \frac{\partial p}{\partial x} + \mu \nabla^2 \dot{u}_x = \rho \ddot{u}_x,$$
$$\rho F_y - \frac{\partial p}{\partial y} + \mu \nabla^2 \dot{u}_y = \rho \ddot{u}_y, \quad (3)$$
$$\rho F_z - \frac{\partial p}{\partial z} + \mu \nabla^2 \dot{u}_z = \rho \ddot{u}_z.$$

This equations form should not depend of any displacement and rotation of the coordinate system. Therefore we should impose any requirements on the variables because "*…not any three functions* $f_i(x,y,z)$ *form a Vector field*" [2, p. 46 http://s013.radikal.ru/i324/1011/c2/55defcd75f63.jpg]. For instance, the gradient of scalar function does not form the vector field [3, p. 30]. Note these little-known representations of university textbooks [2, 3] contradicts to long-established traditional representations of the other textbooks. For instance in [1, p. 120], we can read "*…any three quantities* P, Q, R *can be treated as vector components*" and probably therefore Vector field can be constructed out of scalar fields using the gradient operator [4, p. 316; http://en.wikipedia.org/wiki/Vector_field]. As we can see from [2, p. 46; 3, p. 30] these classical representations are doubtful. We will prove later that classical representations in [1, p. 120] are wrong.

2. Unknown properties of vector function

To prove that not any three functions form a vector field we use the vector lines (streamlines) equations [1, p. 41; 2, p. 57; 3, p. 318]

$$\frac{dx}{\dot{u}_x} = \frac{dy}{\dot{u}_y} = \frac{dz}{\dot{u}_z} = d\varsigma. \quad (4)$$

Here, the symbols d are differentials of x, y, z and ς (scalar parameter) for fix time $t = \bar{t}$.

Equations (4) can now be written in the form

$$\varsigma = \int \frac{1}{\dot{u}_x} dx + C_x \Rightarrow \varsigma = F_x(x, y, z, \bar{t}),$$
$$\varsigma = \int \frac{1}{\dot{u}_y} dy + C_y \Rightarrow \varsigma = F_y(x, y, z, \bar{t}),$$
$$\varsigma = \int \frac{1}{\dot{u}_z} dz + C_z \Rightarrow \varsigma = F_z(x, y, z, \bar{t}). \quad (5)$$

Three expressions $\varsigma = F_i(x, y, z, \bar{t})$ can be considered as any algebraic equations system. This system can be solved in such form

$x = x(\varsigma,\bar{t}),\ y = y(\varsigma,\bar{t}),\ z = z(\varsigma,\bar{t})$. After substitution of these expressions into $\dot{\vec{u}} = \dot{\vec{u}}(x,y,z,\bar{t})$ we will obtain $\dot{u}_i = \dot{u}_i(\varsigma,\bar{t})$.

Now, three obvious equalities can be written

$$\int \frac{1}{\dot{u}_x}dx + C_x = \int \frac{1}{\dot{u}_y}dy + C_y,\ \int \frac{1}{\dot{u}_x}dx + C_x = \int \frac{1}{\dot{u}_z}dz + C_z,\ \int \frac{1}{\dot{u}_y}dy + C_y = \int \frac{1}{\dot{u}_z}dz + C_z.$$

From these equalities we can see that functions $\dot{u}_i = \dot{u}_i(x,y,z,\bar{t})$ cannot be any. This result confirms that "*…not any three functions* $f_i(x,y,z)$ *form a vector field*" [2, p. 46].

From (4) represented in the form

$$\frac{dx}{d\varsigma} = \dot{u}_x(x,y,z,\bar{t}),$$
$$\frac{dy}{d\varsigma} = \dot{u}_y(x,y,z,\bar{t}),$$
$$\frac{dz}{d\varsigma} = \dot{u}_z(x,y,z,\bar{t}) \qquad (6)$$

we can receive the same result $\dot{u}_i = \dot{u}_i(\varsigma,\bar{t})$ or simply $\dot{u}_i = \dot{u}_i(\varsigma)$ without any proof. As well known from [4, p. 65-73], the solutions of such equations can be written so $x = x(\varsigma,\bar{t}),\ y = y(\varsigma,\bar{t}),\ z = z(\varsigma,\bar{t})$. This representation is well known as vector function of a scalar argument [5, p. 514] or a vector-valued function.

We can now state the more detailed properties of functions forming the vector field. To define these properties we use partial derivatives of a composite function, which can be given only one auxiliary variable [4, p. 644]. The velocity vector $\dot{\vec{u}} = \dot{\vec{u}}(x,y,z,\bar{t})$ is such composite function $\dot{\vec{u}} = \dot{\vec{u}}\left(\varsigma(x,y,z,\bar{t})\bar{t}\right)$. We take into account that basic properties of the derivatives are maintained for the vectors [5, p. 516; 6, p. 79]. Then

$$\frac{\partial \dot{\vec{u}}}{\partial x_i} = \frac{\partial \dot{\vec{u}}}{\partial \varsigma}\frac{\partial \varsigma}{\partial x_i},(x_i = x,y,z). \qquad (7)$$

Note that formulas (7) can be written explicitly concerning $\frac{\partial \dot{\vec{u}}}{\partial \varsigma}$. Therefore this common factor can be eliminated. As a result we have

$$\frac{\partial \dot{\vec{u}}}{\partial x_i} = \frac{\partial \dot{\vec{u}}}{\partial x_j}\frac{\partial \varsigma/\partial x_i}{\partial \varsigma/\partial x_j}. \qquad (8)$$

In component form formulas (8) look like

$$\frac{\partial \dot{u}_x}{\partial x_i} = \frac{\partial \dot{u}_x}{\partial x_j}\frac{\partial \varsigma / \partial x_i}{\partial \varsigma / \partial x_j}, \quad \frac{\partial \dot{u}_y}{\partial x_i} = \frac{\partial \dot{u}_y}{\partial x_j}\frac{\partial \varsigma / \partial x_i}{\partial \varsigma / \partial x_j}, \quad \frac{\partial \dot{u}_z}{\partial x_i} = \frac{\partial \dot{u}_z}{\partial x_j}\frac{\partial \varsigma / \partial x_i}{\partial \varsigma / \partial x_j}. \tag{9}$$

Note that relations (9) can be written explicitly concerning $\frac{\partial \varsigma / \partial x_i}{\partial \varsigma / \partial x_j}$. This common factor can be eliminated. Thus, we obtain

$$\frac{\partial \dot{u}_x}{\partial x_i}\frac{\partial \dot{u}_y}{\partial x_j} = \frac{\partial \dot{u}_x}{\partial x_j}\frac{\partial \dot{u}_y}{\partial x_i}, \quad \frac{\partial \dot{u}_x}{\partial x_i}\frac{\partial \dot{u}_z}{\partial x_j} = \frac{\partial \dot{u}_x}{\partial x_j}\frac{\partial \dot{u}_z}{\partial x_i}, \quad \frac{\partial \dot{u}_y}{\partial x_i}\frac{\partial \dot{u}_z}{\partial x_j} = \frac{\partial \dot{u}_y}{\partial x_j}\frac{\partial \dot{u}_z}{\partial x_i}. \tag{9*}$$

This result doesn't mean that equalities (9*) are independent, rather it means that third equality implies from first and second equalities.

Important note (extraordinary). As we can see, equalities (9*) are additional differential equations. These equations are properties (obligatory requirements) of three functions forming the vector field. Therefore, all vector PDE solutions should satisfy these requirements. Otherwise without requirements (9*), all "exact solutions" are not solutions of the vector equations. For instance, NSE solutions can make sense only if $\operatorname{rot}\dot{\vec{u}} \neq 0$ because from (9*) we have

$$\frac{\partial \dot{u}_i}{\partial x_i}\frac{\partial \dot{u}_j}{\partial x_j} = \frac{\partial \dot{u}_i}{\partial x_j}\frac{\partial \dot{u}_j}{\partial x_i}.$$

In the case $\operatorname{rot}\dot{\vec{u}} = 0$ we can see that

$$\frac{\partial \dot{u}_i}{\partial x_j} = \frac{\partial \dot{u}_j}{\partial x_i} \Rightarrow \frac{\partial \dot{u}_i}{\partial x_i}\frac{\partial \dot{u}_j}{\partial x_j} = \left(\frac{\partial \dot{u}_i}{\partial x_j}\right)^2 \Rightarrow \frac{\partial \dot{u}_i}{\partial x_i}\frac{\partial \dot{u}_j}{\partial x_j} \geq 0 .$$

Let's consider these two cases. For first case we have

$$\frac{\partial \dot{u}_i}{\partial x_i}\frac{\partial \dot{u}_j}{\partial x_j} > 0 \Rightarrow \frac{\partial \dot{u}_x}{\partial x} > 0, \ \frac{\partial \dot{u}_y}{\partial y} > 0, \ \frac{\partial \dot{u}_z}{\partial z} > 0 \ \text{ or } \ \frac{\partial \dot{u}_x}{\partial x} < 0, \ \frac{\partial \dot{u}_y}{\partial y} < 0, \ \frac{\partial \dot{u}_z}{\partial z} < 0.$$

As follows from continuity equation (2), such case is impossible.

Let's consider next case

$$\frac{\partial \dot{u}_i}{\partial x_i}\frac{\partial \dot{u}_j}{\partial x_j} = 0 \Rightarrow \frac{\partial \dot{u}_x}{\partial x} = 0, \ \frac{\partial \dot{u}_y}{\partial y} = 0, \ \frac{\partial \dot{u}_z}{\partial z} = 0.$$

For all directions of system of coordinates, this case is impossible. Thus, the requirements $\mathrm{rot}\,\dot{\vec{u}} = 0$, $\mathrm{div}\,\dot{\vec{u}} = 0$ are inconsistent. Therefore vector fields cannot be constructed out of scalar fields using the gradient operator [7, p. 149-153], and so-called Laplacian field [7, p. 214] is not a true vector field. This result confirms a conclusion in [3, p.30] and the little-known proof about impossibility of irrotational velocity field in other university textbook [8, p. 100-101].

Other unpleasant thing we can see for Euler equations (fluid dynamics)

$$\rho\vec{F} - \mathrm{grad}\, p = \rho\,\mathrm{div}\,\ddot{\vec{u}}$$

Note that such equations have no sense because $\mathrm{grad}\, p$ is not a vector. Therefore well-known Helmholtz theorems about vorticities [1, p. 332; 2, p. 115] and numerous Laplacian field researches require revision. The NSE have no such lacks because $(-\mathrm{grad}\, p + \mu\nabla^2\dot{\vec{u}})$ is the vector. However, all so called exact solutions of the Navier–Stokes equations need attentive checking. As we see from (9*), the Couette flow and other simple solutions are exact solutions.

3. Complete equations system

The complete system (as system for vector field completion) may now be written in such form:

$$\rho F_x - \frac{\partial p}{\partial x} + \mu\nabla^2\dot{u}_x = \rho\left(\frac{\partial \dot{u}_x}{\partial t} + \dot{u}_x\frac{\partial \dot{u}_x}{\partial x} + \dot{u}_y\frac{\partial \dot{u}_x}{\partial y} + \dot{u}_z\frac{\partial \dot{u}_x}{\partial z}\right),$$

$$\rho F_y - \frac{\partial p}{\partial y} + \mu\nabla^2\dot{u}_y = \rho\left(\frac{\partial \dot{u}_y}{\partial t} + \dot{u}_x\frac{\partial \dot{u}_y}{\partial x} + \dot{u}_y\frac{\partial \dot{u}_y}{\partial y} + \dot{u}_z\frac{\partial \dot{u}_y}{\partial z}\right),$$

$$\rho F_z - \frac{\partial p}{\partial z} + \mu\nabla^2\dot{u}_z = \rho\left(\frac{\partial \dot{u}_z}{\partial t} + \dot{u}_x\frac{\partial \dot{u}_z}{\partial x} + \dot{u}_y\frac{\partial \dot{u}_z}{\partial y} + \dot{u}_z\frac{\partial \dot{u}_z}{\partial z}\right),$$

$$\frac{\partial \dot{u}_x}{\partial x} + \frac{\partial \dot{u}_y}{\partial y} + \frac{\partial \dot{u}_z}{\partial z} = 0,$$

$$\frac{\partial \dot{u}_x}{\partial x_i}\frac{\partial \dot{u}_y}{\partial x_j} = \frac{\partial \dot{u}_x}{\partial x_j}\frac{\partial \dot{u}_y}{\partial x_i},$$

$$\frac{\partial \dot{u}_x}{\partial x_i}\frac{\partial \dot{u}_z}{\partial x_j} = \frac{\partial \dot{u}_x}{\partial x_j}\frac{\partial \dot{u}_z}{\partial x_i},$$

$$\frac{\partial \dot{u}_y}{\partial x_i}\frac{\partial \dot{u}_z}{\partial x_j} = \frac{\partial \dot{u}_y}{\partial x_j}\frac{\partial \dot{u}_z}{\partial x_i}. \qquad (10)$$

Note that last equality follows from two above. However the true utility of this overdetermined formulation is seen when we try to guess the form of NSE solutions. Let's recollect that we can not receive true solutions of vector equations without these last equalities.

4. Example

As we well know the divergence of a vector field on Euclidean space is a scalar function (or scalar field). Therefore, let's calculate the divergence of acceleration vector $\operatorname{div}\ddot{\vec{u}}$. Then after some transformations

$$\operatorname{div}\ddot{\vec{u}} = \frac{\partial \ddot{u}_x}{\partial x} + \frac{\partial \ddot{u}_y}{\partial y} + \frac{\partial \ddot{u}_z}{\partial z} = \frac{\partial}{\partial t}\operatorname{div}\dot{\vec{u}} + \dot{u}_x \frac{\partial}{\partial x}\operatorname{div}\dot{\vec{u}} + \dot{u}_y \frac{\partial}{\partial y}\operatorname{div}\dot{\vec{u}} + \dot{u}_z \frac{\partial}{\partial z}\operatorname{div}\dot{\vec{u}} +$$
$$+\left[\left(\frac{\partial \dot{u}_x}{\partial x}\right)^2 + \left(\frac{\partial \dot{u}_y}{\partial y}\right)^2 + \left(\frac{\partial \dot{u}_z}{\partial z}\right)^2 + 2\left(\frac{\partial \dot{u}_x}{\partial y}\frac{\partial \dot{u}_y}{\partial x} + \frac{\partial \dot{u}_y}{\partial z}\frac{\partial \dot{u}_z}{\partial y} + \frac{\partial \dot{u}_x}{\partial z}\frac{\partial \dot{u}_z}{\partial x}\right)\right]. \qquad (11)$$

Now we restrict attention to the right hand side of (11). This formula can be written as follows

$$\operatorname{div}\ddot{\vec{u}} = \frac{d}{dt}\operatorname{div}\dot{\vec{u}} + (\operatorname{div}\dot{\vec{u}})^2$$

if such equality is true

$$\left(\frac{\partial \dot{u}_x}{\partial x}\right)^2 + \left(\frac{\partial \dot{u}_y}{\partial y}\right)^2 + \left(\frac{\partial \dot{u}_z}{\partial z}\right)^2 + 2\left(\frac{\partial \dot{u}_x}{\partial y}\frac{\partial \dot{u}_y}{\partial x} + \frac{\partial \dot{u}_y}{\partial z}\frac{\partial \dot{u}_z}{\partial y} + \frac{\partial \dot{u}_x}{\partial z}\frac{\partial \dot{u}_z}{\partial x}\right) = (\operatorname{div}\dot{\vec{u}})^2. \qquad (11^*)$$

The realization of (11*) require such equality:

$$\frac{\partial \dot{u}_x}{\partial y}\frac{\partial \dot{u}_y}{\partial x} + \frac{\partial \dot{u}_y}{\partial z}\frac{\partial \dot{u}_z}{\partial y} + \frac{\partial \dot{u}_x}{\partial z}\frac{\partial \dot{u}_z}{\partial x} = \frac{\partial \dot{u}_x}{\partial x}\frac{\partial \dot{u}_y}{\partial y} + \frac{\partial \dot{u}_y}{\partial y}\frac{\partial \dot{u}_z}{\partial z} + \frac{\partial \dot{u}_x}{\partial x}\frac{\partial \dot{u}_z}{\partial z}. \qquad (12)$$

Let's substitute $x_i = x, y, z, \ (x_j \neq x_i)$ into (9*). Then

$$\frac{\partial \dot{u}_x}{\partial x}\frac{\partial \dot{u}_y}{\partial y} = \frac{\partial \dot{u}_x}{\partial y}\frac{\partial \dot{u}_y}{\partial x}, \frac{\partial \dot{u}_x}{\partial x}\frac{\partial \dot{u}_z}{\partial z} = \frac{\partial \dot{u}_x}{\partial z}\frac{\partial \dot{u}_z}{\partial x}, \frac{\partial \dot{u}_y}{\partial y}\frac{\partial \dot{u}_z}{\partial z} = \frac{\partial \dot{u}_y}{\partial z}\frac{\partial \dot{u}_z}{\partial y}.$$

These last equalities are the necessary conditions for equalities (11*), (12). In this case we have $\text{div}\ddot{\vec{u}} = \frac{d}{dt}\text{div}\dot{\vec{u}} + (\text{div}\dot{\vec{u}})^2$. Therefore $\text{div}\ddot{\vec{u}}$ is a scalar function.

7. Conclusion

Without equalities (9*), system (2), (3) is determined not completely. The completely vector system looks like (10). Let's recollect that we can not receive true NSE solutions without last equalities. Therefore, all so called "3D NSE exact solutions" [9, 10] need attentive checking. However the exact solutions of vector equations (10) and other vector PDE can appear wrong in case of not co-ordinated or not smoothed initial and boundary conditions (about this problem for wave equation [11, p. 63-83]).

Reference

[1] Sedov L.I. (1970) Continuum Mechanics, v. 1. Textbook, Science, Moscow (djvu) http://eqworld.ipmnet.ru/ru/library/books/Sedov_MSS_t1_1970ru.djvu

[2] Lojtsjansky L.G. (1970). Fluid and Gas Mechanics. Manual, Science, Moscow (djvu) http://eqworld.ipmnet.ru/ru/library/books/Lojcyanskij1950ru.djvu

[3] Dubrovin B.A., Novikov S. P., Fomenko A.T. (1986) Modern geometry**.** Manual, Science, Moscow, (djvu) http://www.newlibrary.ru/download/dubrovin_b_a___novikov_s_p___fomenko_a_t_/sovremennaja_geometrija_(metody_i_prilozhenija).html

[4] Smirnov V.I. (1974) Higher Mathematics Course, v. 2 (21th edition). Textbook, Science, Moscow (djvu) http://eqworld.ipmnet.ru/ru/library/books/Smirnov_t2_1974ru.djvu

[5] Vygodsky M.J. (1977) Manual on Higher Mathematics (12th edition). Science (djvu) http://eqworld.ipmnet.ru/ru/library/books/Vygodskij1977ru.djvu

[6] Kochin N.E. (1965) Vector Calculus and beginnings of Tensors Calculus (9th edition). Science, Moscow (djvu) http://eqworld.ipmnet.ru/ru/library/books/Kochin1965ru.djvu

[7] Borisenko A.I., Tarapov I.E. (1966) Vector Analysis and beginnings of Tensors Calculus (3th edition). Manual, Higher school, Moscow (djvu) http://eqworld.ipmnet.ru/ru/library/books/BorisenkoTarapov1966ru.djvu

[8] Slezkin N.A. (1955) Dynamics of the Viscous Incompressible Fluids. Textbook, State publishing house, Moscow (djvu) http://eqworld.ipmnet.ru/ru/library/books/Slezkin1955ru.djvu

[9] Aristov S.N., Poljanin A. D. (2009) Exact solutions of three-dimensional non-stationary Navier-Stokes Equations, *RAS Reports* , v. 427, № 1, p. 35–40, http://eqworld.ipmnet.ru/ru/solutions/interesting/aristov_polyanin2009.pdf

[10] Jormakka, J. (2010), Solutions to three-dimensional Navier-Stokes equations for incompressible fluids, *Electronic Journal of Differential Equations* **2010** (93): 1–14, http://emis.impa.br/EMIS/journals/EJDE/Monographs/Volumes/2010/93/jormakka.pdf

[11] Kozachok A. (2009) Paradoxes of Continuum Mechanics and contiguous fields of knowledge. Section 1. New approaches to statements and solutions of some classical Mathematical Physics problems . Manual, VDM Verlag Dr. Muller, ISBN 978-3639135954 http://www.shelfari.com/books/13880604/PARADOXES-of-CONTINUUM-MECHANICS-and-contiguous-fields-of-knowle