Paradoxes of Helmholtz decomposition and its Inverse Problem

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Abstract. Fundamental theorem of vector calculus is based on the Helmholtz decomposition (sometimes called Helmholtz-Hodge decomposition) of any vector field into an irrotational part and a solenoidal part. In this paper we prove that Helmholtz decomposition and its Inverse problem are wrong and require major revision. As we well know from differential forms theory the Hodge decomposition is related to the Helmholtz decomposition. Therefore Hodge decomposition requires revision also. For Fundamental theorem of vector calculus (therefore for Fundamental Inverse problem) we establish a new formula which completely corresponds to Navier–Stokes equations and Lame (also called Navier) equations (equations of linear elasticity). This paper written in a way that gives insight to mathematicians who may not be experts in this topic (comparison of information in different textbooks and improve of information in Modern Geometry by clear counterexample).

Key Words and Phrases: vector field, gradient, Navier – Stokes equations, Lame (Navier) equations, Euler equations, Helmholtz decomposition, inverse problem, Hodge decomposition, antisymmetric tensor.
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1. Introduction

From different textbooks we well know that a vector field can be constructed with both a specified divergence and a specified curl, and if vector field also vanishes at infinity, it is uniquely specified by its divergence and curl. This well-known approach is used in mathematics and physics (for example a derivation of Maxwell's equations) and can be considered as Inverse problem of Fundamental theorem of vector calculus and its applications in many branches of science.

Note that Fundamental theorem of vector calculus (also called Helmholtz decomposition) states that any sufficiently smooth, rapidly decaying vector field in three dimensions $\mathbf{F}$ can be constructed with the sum of an irrotational (curl-free) vector field $\operatorname{grad}\varphi$ and a solenoidal (divergence-free) vector field $\operatorname{rot}\mathbf{A}$ ($\varphi, \mathbf{A}$ are so called scalar potential φ and a vector potential $\mathbf{A}$)

$$\mathbf{F} = -\operatorname{grad}\psi + \operatorname{rot}\mathbf{A} \Rightarrow \mathbf{F} = \operatorname{grad}\varphi + \operatorname{rot}\mathbf{A}. \qquad (1)$$

Therefore each Inverse problem of this theorem can be written as follows if left-hand side is specified

$$\operatorname{grad}\varphi + \operatorname{rot}\mathbf{A} = \mathbf{F} \ . \qquad (1^*)$$

But, in this university textbook we can read [1, p. 15] « … *under co-ordinate change the gradient of function transforms differently from a vector* ». Therefore the gradient of scalar function does not construct a true vector field.

The next unpleasant things we can establish for such well-known classical rule. In mathematics and physics the rot (curl) is an operation which takes any vector field $\mathbf{A}$ and produces another vector field $\operatorname{rot}\mathbf{A}$. However it is known that $\operatorname{rot}\mathbf{A}$ (also called pseudovector) is equivalent to the Antisymmetric Tensor [2, p. 183]. In that case under co-ordinate change the Antisymmetric Tensor components should transform differently from the true vector components. Therefore the author of textbook [3, p. 50] paid attention that «… *pseudovector … from the point of view of its vector product on other true vector is equivalent to antisymmetric tensor, but as the vector cannot be equal to tensor*». In other words, the Antisymmetric Tensor can be found as decomposition of any rank-2 tensor. For example [3, p.62, 63] a tensor of partial derivates of velocity vector $\dot{\vec{u}}$ can be written as a sum of symmetric and antisymmetric parts:

$$\begin{pmatrix} \frac{\partial \dot{u}_x}{\partial x} & \frac{\partial \dot{u}_x}{\partial y} & \frac{\partial \dot{u}_x}{\partial z} \\ \frac{\partial \dot{u}_y}{\partial x} & \frac{\partial \dot{u}_y}{\partial y} & \frac{\partial \dot{u}_y}{\partial z} \\ \frac{\partial \dot{u}_z}{\partial x} & \frac{\partial \dot{u}_z}{\partial y} & \frac{\partial \dot{u}_z}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial \dot{u}_x}{\partial x} & \frac{1}{2}\left(\frac{\partial \dot{u}_x}{\partial y} + \frac{\partial \dot{u}_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial \dot{u}_x}{\partial z} + \frac{\partial \dot{u}_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial \dot{u}_y}{\partial x} + \frac{\partial \dot{u}_x}{\partial y}\right) & \frac{\partial \dot{u}_y}{\partial y} & \frac{1}{2}\left(\frac{\partial \dot{u}_y}{\partial z} + \frac{\partial \dot{u}_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial \dot{u}_z}{\partial x} + \frac{\partial \dot{u}_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial \dot{u}_z}{\partial y} + \frac{\partial \dot{u}_y}{\partial z}\right) & \frac{\partial \dot{u}_z}{\partial z} \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial \dot{u}_x}{\partial y} - \frac{\partial \dot{u}_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial \dot{u}_x}{\partial z} - \frac{\partial \dot{u}_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial \dot{u}_y}{\partial x} - \frac{\partial \dot{u}_x}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial \dot{u}_y}{\partial z} - \frac{\partial \dot{u}_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial \dot{u}_z}{\partial x} - \frac{\partial \dot{u}_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial \dot{u}_z}{\partial y} - \frac{\partial \dot{u}_y}{\partial z}\right) & 0 \end{pmatrix}$$

As we can see the symmetric part is a velocity deformation tensor $(\dot{\varepsilon}_{ij})$, $\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ji}$. The antisymmetric part is Antisymmetric Tensor which contains the so called pseudovector components $\operatorname{rot}\dot{\vec{u}}$. Therefore above expression can be written so

$$\begin{pmatrix} \frac{\partial \dot{u}_x}{\partial x} & \frac{\partial \dot{u}_x}{\partial y} & \frac{\partial \dot{u}_x}{\partial z} \\ \frac{\partial \dot{u}_y}{\partial x} & \frac{\partial \dot{u}_y}{\partial y} & \frac{\partial \dot{u}_y}{\partial z} \\ \frac{\partial \dot{u}_z}{\partial x} & \frac{\partial \dot{u}_z}{\partial y} & \frac{\partial \dot{u}_z}{\partial z} \end{pmatrix} = \begin{pmatrix} \dot{\varepsilon}_{xx} & \dot{\varepsilon}_{xy} & \dot{\varepsilon}_{xz} \\ \dot{\varepsilon}_{yx} & \dot{\varepsilon}_{yy} & \dot{\varepsilon}_{yz} \\ \dot{\varepsilon}_{zx} & \dot{\varepsilon}_{zy} & \dot{\varepsilon}_{zz} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{2}\mathrm{rot}_z \dot{\vec{u}} & \frac{1}{2}\mathrm{rot}_y \dot{\vec{u}} \\ \frac{1}{2}\mathrm{rot}_z \dot{\vec{u}} & 0 & -\frac{1}{2}\mathrm{rot}_x \dot{\vec{u}} \\ -\frac{1}{2}\mathrm{rot}_y \dot{\vec{u}} & \frac{1}{2}\mathrm{rot}_x \dot{\vec{u}} & 0 \end{pmatrix}.$$

The above results show very well that the theory requiring Helmholtz decomposition (1) and its inverse problems (1*) must be false. Such examples are known.

Counterexample. As we well know the divergence of any vector field on Euclidean space is a scalar field. Therefore as an example let's calculate the divergence of an acceleration vector $\mathrm{div}\ddot{\vec{u}}$ (fluid flow). In expanded form the acceleration vector components $\ddot{\vec{u}}$ can be written so [2, p. 39]

$$\ddot{u}_x = \frac{d\dot{u}_x}{dt} = \frac{\partial \dot{u}_x}{\partial t} + \dot{u}_x \frac{\partial \dot{u}_x}{\partial x} + \dot{u}_y \frac{\partial \dot{u}_x}{\partial y} + \dot{u}_z \frac{\partial \dot{u}_x}{\partial z},$$

$$\ddot{u}_y = \frac{d\dot{u}_y}{dt} = \frac{\partial \dot{u}_y}{\partial t} + \dot{u}_x \frac{\partial \dot{u}_y}{\partial x} + \dot{u}_y \frac{\partial \dot{u}_y}{\partial y} + \dot{u}_z \frac{\partial \dot{u}_y}{\partial z},$$

$$\ddot{u}_z = \frac{d\dot{u}_z}{dt} = \frac{\partial \dot{u}_z}{\partial t} + \dot{u}_x \frac{\partial \dot{u}_z}{\partial x} + \dot{u}_y \frac{\partial \dot{u}_z}{\partial y} + \dot{u}_z \frac{\partial \dot{u}_z}{\partial z}.$$

After taking an operator div we have

$$\mathrm{div}\ddot{\vec{u}} = \frac{\partial \ddot{u}_x}{\partial x} + \frac{\partial \ddot{u}_y}{\partial y} + \frac{\partial \ddot{u}_z}{\partial z} = \frac{\partial}{\partial t}\mathrm{div}\dot{\vec{u}} + \dot{u}_x \frac{\partial}{\partial x}\mathrm{div}\dot{\vec{u}} + \dot{u}_y \frac{\partial}{\partial y}\mathrm{div}\dot{\vec{u}} + \dot{u}_z \frac{\partial}{\partial z}\mathrm{div}\dot{\vec{u}} +$$
$$+\left[\left(\frac{\partial \dot{u}_x}{\partial x}\right)^2 + \left(\frac{\partial \dot{u}_y}{\partial y}\right)^2 + \left(\frac{\partial \dot{u}_z}{\partial z}\right)^2 + 2\left(\frac{\partial \dot{u}_x}{\partial y}\frac{\partial \dot{u}_y}{\partial x} + \frac{\partial \dot{u}_y}{\partial z}\frac{\partial \dot{u}_z}{\partial y} + \frac{\partial \dot{u}_x}{\partial z}\frac{\partial \dot{u}_z}{\partial x}\right)\right]. \quad (2)$$

This formula can be written as

$$\mathrm{div}\ddot{\vec{u}} = \frac{d}{dt}\mathrm{div}\dot{\vec{u}} + (\mathrm{div}\dot{\vec{u}})^2 -$$
$$-2\left[\left(\frac{\partial \dot{u}_x}{\partial x}\frac{\partial \dot{u}_y}{\partial y} - \frac{\partial \dot{u}_x}{\partial y}\frac{\partial \dot{u}_y}{\partial x}\right) + \left(\frac{\partial \dot{u}_x}{\partial x}\frac{\partial \dot{u}_z}{\partial z} - \frac{\partial \dot{u}_x}{\partial z}\frac{\partial \dot{u}_z}{\partial x}\right) + \left(\frac{\partial \dot{u}_y}{\partial y}\frac{\partial \dot{u}_z}{\partial z} - \frac{\partial \dot{u}_y}{\partial z}\frac{\partial \dot{u}_z}{\partial y}\right)\right].$$

As we can see

$$\mathrm{div}\ddot{\vec{u}} = \frac{d}{dt}\mathrm{div}\dot{\vec{u}} + (\mathrm{div}\dot{\vec{u}})^2 \quad (3)$$

if and only if such equality is true

$$\left(\frac{\partial \dot{u}_x}{\partial x}\right)^2+\left(\frac{\partial \dot{u}_y}{\partial y}\right)^2+\left(\frac{\partial \dot{u}_z}{\partial z}\right)^2+2\left(\frac{\partial \dot{u}_x}{\partial y}\frac{\partial \dot{u}_y}{\partial x}+\frac{\partial \dot{u}_y}{\partial z}\frac{\partial \dot{u}_z}{\partial y}+\frac{\partial \dot{u}_x}{\partial z}\frac{\partial \dot{u}_z}{\partial x}\right)=(\mathrm{div}\,\dot{\vec{u}})^2 \quad (4)$$

and for example if such additional conditions are satisfied

$$\frac{\partial \dot{u}_x}{\partial x}\frac{\partial \dot{u}_y}{\partial y}=\frac{\partial \dot{u}_x}{\partial y}\frac{\partial \dot{u}_y}{\partial x},\frac{\partial \dot{u}_x}{\partial x}\frac{\partial \dot{u}_z}{\partial z}=\frac{\partial \dot{u}_x}{\partial z}\frac{\partial \dot{u}_z}{\partial x},\frac{\partial \dot{u}_y}{\partial y}\frac{\partial \dot{u}_z}{\partial z}=\frac{\partial \dot{u}_y}{\partial z}\frac{\partial \dot{u}_z}{\partial y}\;. \quad (5)$$

Note that equality (4) can make sense only if $\mathrm{rot}\,\dot{\vec{u}}\neq 0$. In the case $\mathrm{rot}\,\dot{\vec{u}}=0$ all terms (in brackets) of left-hand side (4) are positive and $\mathrm{div}\,\dot{\vec{u}}=0$ is impossible. As we well know for $\dot{\vec{u}}=\mathrm{grad}\,\varphi \;\Rightarrow \nabla^2\varphi=0$, if $\mathrm{rot}\,\dot{\vec{u}}=0$, $\mathrm{div}\,\dot{\vec{u}}=0$. Thus vector fields cannot be constructed out of scalar fields using the gradient operator $\dot{\vec{u}}=\mathrm{grad}\,\varphi$. This result improves above information in [1, p. 15].

2. Elimination of contradictions

For elimination of above contradictions the Fundamental theorem of vector calculus can be written as follows:

$$\mathbf{F}=\mathrm{grad}\,\varphi+\mathrm{rot}\,\mathrm{rot}\,\bar{\mathbf{A}}\;. \quad (6)$$

This formula completely corresponds to the Navier–Stokes equations (NSE) for incompressible fluids. As we well know

$$\nabla^2\dot{\vec{u}}=\mathrm{grad}\,\mathrm{div}\,\dot{\vec{u}}-\mathrm{rotrot}\,\dot{\vec{u}}\,. \quad (7)$$

Therefore from NSE we obtain (if $\mathrm{div}\,\dot{\vec{u}}=0$)

$$\rho\ddot{\vec{u}}=-\mathrm{grad}\,p+\mu\nabla^2\dot{\vec{u}}+\rho\vec{F}\Rightarrow\rho(\vec{F}-\ddot{\vec{u}})=\mathrm{grad}\,p+\mathrm{rotrot}\,\mu\dot{\vec{u}}\;. \quad (8)$$

Here, $\bar{\mathbf{A}}$ - true vector, $\vec{F}=\vec{F}_1+\vec{F}_2+...$ - vectors sum of a given, externally applied forces (e.g. gravity $\vec{F}_1$, magnetic $\vec{F}_2$ and other), p - pressure (scalar function), $\dot{\vec{u}}$ - velocity vector, $\ddot{\vec{u}}=d\dot{\vec{u}}/dt$ - acceleration vector, ρ - density (const), μ - viscosity (const), ∇^2 - Laplace operator.

As we can see equation (6) and equation (8) coincide. Hence there is no reason to say that the theory requiring (6) must be false. As follows from NSE this sum:

$$-\mathrm{grad}\, p + \mu\nabla^2\dot{\vec{u}} = -(\mathrm{grad}\, p + \mathrm{rot}\,\mathrm{rot}\,\mu\dot{\vec{u}})$$

can construct a true vector field.

Note that we will obtain formula (6) also after similar transformation of the Navier–Stokes for a compressible fluid and after transformation of the Lame equations (equations of linear elasticity). According [4, p. 300] the Lame equations (also known as Navier eq.) can be written so

$$\rho\ddot{\vec{u}} = (\lambda + 2\bar{\mu})\,\mathrm{grad}\,\mathrm{div}\,\vec{u} - \bar{\mu}\,\mathrm{rot}\,\mathrm{rot}\,\vec{u} + \rho\vec{F} \quad . \qquad (9)$$

After minor transformation of (9) we can see that forms of equations (6), (8) and below equations completely coincide

$$\rho(\vec{F} - \ddot{\vec{u}}) = \mathrm{grad}\,\mathrm{div}\left(-(\lambda + 2\bar{\mu})\vec{u}\right) + \mathrm{rot}\,\mathrm{rot}\,\bar{\mu}\vec{u}\,.$$

Here $\vec{u}$ - displacement vector, λ, $\bar{\mu}$ - Lame constants.

Notation. Formula (7) can be written as

$$-\nabla^2\dot{\vec{u}} = -\mathrm{grad}\,\mathrm{div}\,\dot{\vec{u}} + \mathrm{rot}\,\mathrm{rot}\,\dot{\vec{u}} \Rightarrow -\nabla^2\dot{\vec{u}} = \mathrm{grad}\,\varphi + \mathrm{rot}\,\mathrm{rot}\,\bar{\mathbf{A}}, \qquad (7^*)$$
$$\varphi = -\mathrm{div}\,\dot{\vec{u}}, \quad \bar{\mathbf{A}} = \dot{\vec{u}}.$$

After comparison of (7*) and (6) we obtain $-\nabla^2\dot{\vec{u}} = \mathbf{F}$ and therefore $\nabla^2\dot{\vec{u}}$ constructs true vector field ? However such conclusion is wrong **as not each sum $\mathrm{grad}\,\varphi + \mathrm{rot}\,\mathrm{rot}\,\bar{\mathbf{A}}$ can construct true vector fields.** Therefore all *Inverse problems of Helmholtz's theorem require additional conditions for the sum* $\mathrm{grad}\,\varphi + \mathrm{rot}\,\mathrm{rot}\,\bar{\mathbf{A}}$ *which constructs the true vector field* $\mathbf{F}$.

3. Consequences for different areas of science

The vector fields cannot be constructed out of scalar fields using the gradient operator $\dot{\vec{u}} = \mathrm{grad}\,\varphi$. As we well know $\dot{\vec{u}} = \mathrm{grad}\,\varphi \Rightarrow \nabla^2\varphi = 0$, if $\mathrm{rot}\,\dot{\vec{u}} = 0$, $\mathrm{div}\,\dot{\vec{u}} = 0$. Therefore so-called Laplacian field is not a true vector field. Thus, the requirements $\mathrm{rot}\,\dot{\vec{u}} = 0, \mathrm{div}\,\dot{\vec{u}} = 0$ are incompatible for true vector fields. This result confirms the proof about impossibility of irrotational velocity field in this fluid dynamics textbook [5, p. 100-101].

Other unpleasant things we can see for well-known classical equations. For example the Euler equations (fluid dynamics) can be written as follows

$$-\text{grad}\, p = \rho(\text{div}\ddot{\vec{u}} - \vec{F}).$$

Note that such equations have no sense as exact vector equations because left-hand side $\text{grad}\, p$ is not the true vector.

This wrong theorem (more strictly Inverse problem**)** is of great importance in electrostatics (Maxwell's equations) and other areas of mathematical physics. Thus we can continue a list of similar incorrect fundamental equations.

As we well know from differential forms theory the Hodge decomposition is related to the Helmholtz decomposition. Therefore Hodge decomposition requires revision also. As an example for elimination of the Hodge decomposition contradictions the NSE can be written as follows

$$d(\dot{\vec{u}} - \vec{R}) = \frac{1}{\rho}(-\text{grad}\, p + \mu\nabla^2\dot{\vec{u}})dt\,.$$

if vectors of a given, externally applied forces $\vec{F} = d\vec{R}/dt$.

4. Conclusion

The Fundamental Inverse problem of vector calculus is based on the Helmholtz decomposition (sometimes called Helmholtz-Hodge decomposition or Fundamental theorem of vector calculus) of any vector field into a solenoidal part and an irrotational part.

The Helmholtz decomposition is more 100 years old. This theorem and different Inverse problems are used by thousand researchers, covered in hundreds textbooks, and taught to thousands students all over the world. Mathematicians have proved not only an existence but also Uniqueness of Helmholtz decomposition and its Inverse problem. The Helmholtz decomposition used for transformations of the Navier–Stokes and Lame (also called Navier) equations. These transformations included in encyclopedias and different textbooks as counterexamples (for instance Pressure-free velocity formulation). But, from this university textbook [1, p. 15] it is well known for a long time (1979 - first Russian edition) that « … *under co-ordinate change the gradient of function transforms differently from a vector*». Therefore the gradient of scalar function does not construct a true vector field. It means this well-known theorem has appeared total wrong. Hence all inverse problems requiring (1*) must be false. However this important information in Modern Geometry [1, p. 15] is ignored everywhere.

For elimination of mentioned contradictions the Fundamental theorem of vector calculus can be written as follows from (6). This formula completely corresponds to minor transformed Navier–Stokes equations for compressible and incompressible fluids and the Lame equations for an elastic media. Therefore Inverse problem**s** of Helmholtz's theorem require additional conditions for the sum $\text{grad}\,\varphi + \text{rot}\,\text{rot}\,\bar{\mathbf{A}}$ which constructs the true vector field $\mathbf{F}$.

As we well know from differential forms theory the Hodge decomposition is related to the Helmholtz decomposition. Therefore Hodge decomposition requires revision also.

These problems, very unpleasant for many areas of modern science, were discussed here [6].

This paper has been written after considering by Prof. James E. Colliander of another paper submitted to Proceedings of the American Mathematical Society. According to the referee reports "the key observation revolves around issues related to the Hodge decomposition but the author does not place the discussion within that context… If developed in more detail and connected with related literature, this paper might merit appearance in a specialized PDE journal". Therefore I do thank for this important referee reports. Without referee remark about Hodge decomposition this paper would be uncompleted.

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